

B. Tech.
(SEM. I) EXAMINATION, 2007-08

MATHEMATICS-I

Time: 3 Hours

Total Marks: 100

Note: Attempt all the problems. Internal choice are mentioned in every problem.

Q. 1. Attempt any Two parts : (10×2=20)

(a) Define the given values, eigenvectors and the characteristic equation of a square matrix. Find the characteristics equation/polynomail, eigen values and eigen vectors of the matrix :

$$\begin{bmatrix} 2 & 5 & 7 \\ 5 & 3 & 1 \\ 7 & 0 & 2 \end{bmatrix}$$

Ans. Characteristic Equation : Let A be a square matrix. Then the equation

$$|A - \lambda I| = 0$$

is called characteristic equation of matrix A .

Eigen Value : Let A be a square matrix. The values of λ satisfying

$$|A - \lambda I| = 0$$

are called eigen values of matrix A .

Eigen Vectors : Let A be a square matrix. Let λ be an eigen value of A . A non zero column vector x satisfying

$(A - \lambda I)x = 0$ is called eigen vector of matrix A corresponding to eigen value λ .

IInd Part :

Given matrix $A = \begin{bmatrix} 2 & 5 & 7 \\ 5 & 3 & 1 \\ 7 & 0 & 2 \end{bmatrix}$

Characteristics equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 5 & 7 \\ 5 & 3-\lambda & 1 \\ 7 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(3-\lambda)(2-\lambda)] - 5[5(2-\lambda) - 7] + 7[-7(3-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)(6-3\lambda-2\lambda+\lambda^2) - 5(10-5\lambda-7) + 7(-21+7\lambda) = 0$$

$$\Rightarrow (2-\lambda)(6-5\lambda+\lambda^2) - 5(3-5\lambda) - 147 + 49\lambda = 0$$

$$\Rightarrow 12-10\lambda+2\lambda^2-6\lambda+5\lambda^2-\lambda^3-15+25\lambda-147+49\lambda = 0$$

$$\Rightarrow -\lambda^3+7\lambda^2+52\lambda-150 = 0$$

$$\Rightarrow \lambda^3-7\lambda^2-52\lambda+150 = 0$$

which is not further solvable at our level

Q.1. (b) Check the consistancy of the following system of linear non-homogeneous equations and find the solution, if exists :

$$7x_1 + 2x_2 + 3x_3 = 16$$

$$2x_1 + 11x_2 + 5x_3 = 25$$

$$x_1 + 3x_2 + 4x_3 = 13$$

Ans. Given system of equation is

$$7x_1 + 2x_2 + 3x_3 = 16$$

$$2x_1 + 11x_2 + 5x_3 = 25$$

$$x_1 + 3x_2 + 4x_3 = 13$$

Augmented matrix $[A, B] = \left[\begin{array}{ccc|c} 7 & 2 & 3 & 16 \\ 2 & 11 & 5 & 25 \\ 1 & 3 & 4 & 13 \end{array} \right]$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 2 & 11 & 5 & 25 \\ 7 & 2 & 3 & 16 \end{array} \right]$$

$$R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 0 & 5 & -3 & -1 \\ 0 & -19 & -25 & -15 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 7R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 13 \\ 0 & 20 & -12 & -4 \\ 0 & -115 & -55 & -106 \end{array} \right]$$

$$R_2 \rightarrow 4R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & | & 13 \\ 0 & 1 & -37 & | & -79 \\ 0 & -19 & -25 & | & -75 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & | & 13 \\ 0 & 1 & -37 & | & -79 \\ 0 & 0 & -7281 & | & -1576 \end{bmatrix} \quad R_3 \rightarrow R_3 + 19R_2$$

Which is in Cohelon form

Here, Rank of $A = 3$

Rank of $[A, B] = 3$

No. of unknown = 3

: Rank of $[A, B] = \text{Rank of } A = \text{No. of unknown}$

: System has unique soln.

Rewriting the system,

$$1x_1 + 3x_2 + 4x_3 = 13 \quad \dots(1)$$

$$1x_2 - 37x_3 = -79 \quad \dots(2)$$

$$-728x_3 = -1567 \quad \dots(3)$$

From (3),

$$x_3 = \frac{-1576}{-728} = \frac{197}{91}$$

Putting x_3 in (2),

$$x_2 = -79 + 37 \times \frac{197}{91}$$

$$\Rightarrow x_2 = \frac{100}{91}$$

Putting x_2 and x_3 in (1),

$$x_1 = 13 - 3 \times \frac{100}{91} - 4 \times \frac{197}{91}$$

$$\Rightarrow x_1 = \frac{95}{91}$$

Thus

$$x_1 = \frac{95}{91}, x_2 = \frac{100}{91}, x_3 = \frac{197}{91}$$

Q. 1 (c) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1/3 & 1/5 & 1/7 \\ 1/5 & 1/7 & 1/11 \end{bmatrix}$$

Ans. Given matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1/5 & 1/7 & 1/11 \end{bmatrix}$

$$\therefore A = IA$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1/3 & 1/5 & 1/7 \\ 1/5 & 1/7 & 1/11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 5 \\ 1/3 & 1/5 & 1/7 \\ 1/5 & 1/7 & 1/11 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \rightarrow 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & +1 \\ 0 & +1 & 3 \\ 0 & +1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \\ 5 & 0 & 1 \end{bmatrix} A$$

$$R_2 \rightarrow R_2 - \frac{R_1}{3}$$

$$R_3 \rightarrow R_3 - \frac{R_1}{5}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & +1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 30 & -45 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

$$R_2 \rightarrow 45R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 24 \\ 0 & 1 & -3 \\ 0 & 0 & 202 \end{bmatrix} = \begin{bmatrix} -18 & 30 & 0 \\ 30 & -45 & 0 \\ -24 & +3 & 1 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 - \frac{2}{3} R_2$$

$$R_3 \rightarrow R_3 - \frac{1}{103} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{24}{35} \\ 0 & 1 & -\frac{3}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -18 & 30 & 0 \\ 30 & -45 & 0 \\ -4620 & +5775 & 13475 \\ 101 & 202 & 202 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1350}{101} & \frac{1050}{101} & \frac{-4620}{101} \\ 1050 & -6615 & +5775 \\ \frac{101}{202} & \frac{202}{202} & \frac{202}{202} \\ -4620 & +5775 & 13475 \\ \frac{101}{202} & \frac{202}{202} & \frac{202}{202} \end{bmatrix} A$$

$$R_1 \rightarrow R_2 - \frac{24}{35} R_3$$

$$R_2 \rightarrow R_2 + \frac{3}{7} R_3$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1350}{101} & \frac{1050}{101} & \frac{-4620}{101} \\ 1050 & -6615 & 5775 \\ \frac{101}{202} & \frac{202}{202} & \frac{202}{202} \\ -4620 & 5775 & 13475 \\ \frac{101}{202} & \frac{202}{202} & \frac{202}{202} \end{bmatrix}$$

$$= \frac{1}{202} \begin{bmatrix} 2700 & -13230 & -9240 \\ 2100 & -13230 & 5775 \\ -9240 & 5775 & 13475 \end{bmatrix} \text{ Ans.}$$

Q. 2. Attempt any Two parts : (10×2=20)

(a) State Leibnitz theorem for n^{th} differential coefficient of the product of two functions. If $y^{1/m} +$

$y^{-1/m} = 2x$, prove that

$$(x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$$

Ans. Leibnitz's Theorem : For two functions u and v , $(uv)_n = n C_0 u v + n C_1 u_n v_1 + \dots + n C_n u v_n$

2nd part :

$$\frac{1}{y^m} + \frac{1}{y^{-m}} = 2x$$

$$\Rightarrow z + z^{-1} = 2x$$

$$\Rightarrow z^2 + 1 = 2xz$$

$$\text{Put } \frac{1}{y^m} = z$$

$$\Rightarrow z^2 - 2xz + 1 = 0$$

$$\Rightarrow z = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 - 1}}{2}$$

$$\Rightarrow z = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow y^{1/m} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow y = (x \pm \sqrt{x^2 - 1})^m$$

Case - II

$$\text{When } y = (x \pm \sqrt{x^2 - 1})^m$$

$$\Rightarrow y_1 = m (x + \sqrt{x^2 - 1})^{m-1}$$

$$\left(1 + \frac{2x}{2\sqrt{x^2 - 1}}\right)$$

$$\Rightarrow \sqrt{x^2 - 1} y_1 = m (x + \sqrt{x^2 - 1})^{m-1}$$

$$(x + \sqrt{x^2 - 1})$$

$$\Rightarrow \sqrt{x^2 - 1} y_1 = -m (x - \sqrt{x^2 - 1})^m$$

$$\Rightarrow \sqrt{x^2 - 1} y_1 = -m y$$

$$\Rightarrow (x^2 - 1)y_1^2 = m^2 y^2$$

Thus in both cases,

$$(x^2 - 1)y_1^2 = m^2 y^2$$

Differentiating above expression,

$$2x y_1^2 + (x^2 - 1) 2y_1 y_2 = m^2 2y y_1$$

$$\Rightarrow x y_1 + (x^2 - 1) y_2 = m^2 y$$

$$\Rightarrow (x^2 + 1) y_2 + x y_1 - m^2 y = 0 \quad \dots (1)$$

Differentiating (1) n times by Leibnitz's theorem,

$$(x^2 - 1)y_{n+2} + {}^n C_1 (2x) y_{n+1} + {}^n C_2 (2) y_n$$

$$+ [x y_{n+1} + {}^n C_1 (1) y_n] - m^2 y_n = 0.$$

$$\Rightarrow (x^2 - 1)y_{n+2} + 2nx y_{n+1} + n(n-1)y_n$$

$$+ x y_{n+1} - x y_n - m^2 y_n = 0$$

$$\left[\because C_1 = n C_2 = \frac{n(n-1)}{2} \right]$$

$$\Rightarrow (x^2 - 1)y_{n+2} + (2nx + x)y_{n+1}$$

$$+ [n(n-1) + n - m^2] y_n = 0$$

$$\Rightarrow (x^2 - 1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 + m^2)y_n = 0 \quad \text{Hence Proved.}$$

Q. 2 (b) Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

where $u(x, y) = \log_e \left(\frac{x^2 + y^2}{xy} \right)$

Ans. $u = \log \frac{x^2 + y^2}{xy}$

To verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

We shall first calculate $\frac{\partial^2 u}{\partial x \partial y}$

Now $\frac{\partial u}{\partial x} = \log \frac{x^2 + y^2}{xy}$

$\Rightarrow u = \log(x^2 + y^2) - \log x - \log y$

$\Rightarrow \frac{\partial u}{\partial y} = \frac{xy}{x^2 + y^2} - \frac{1}{y}$

$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right)$

$= \frac{\partial}{\partial x} \left[\frac{2y}{x^2 + y^2} - \frac{1}{y} \right] = \frac{2y(2x)}{-(x^2 + y^2)^2}$

$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{-4xy}{(x^2 + y^2)^2} \quad \dots (1)$

Again, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} [\log(x^2 + y^2) - \log x - \log y]$

$= \frac{2x}{x^2 + y^2} - \frac{1}{x}$

$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right)$

$= \frac{\partial}{\partial y} \left(\frac{2x}{x^2 + y^2} - \frac{1}{x} \right)$

$= -2x \left(\frac{+1}{x^2 + y^2} \right)^2 2y$

$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = -\frac{4xy}{(x^2 + y^2)^2} \quad \dots (2)$

From (1) and (2)

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial y \partial x}$$

Q. 2 (c) If $u = x \sin^{-1} \left(\frac{x}{y} \right) + y \sin^{-1} \left(\frac{y}{x} \right)$, find the

value of $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

Ans. $u = x \sin^{-1} \left(\frac{x}{y} \right) + y \sin^{-1} \left(\frac{y}{x} \right)$

$\Rightarrow u(x, y) = x \left[\sin^{-1} \left(\frac{x}{y} \right) + \frac{y}{x} \sin^{-1} \left(\frac{y}{x} \right) \right]$
 $= x^0 f \left(\frac{y}{x} \right)$

$\Rightarrow u$ is a homogeneous function of degree $n = 1$

\therefore By Euler's theorem :

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$

$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u \quad \dots (1)$

Differentiating (1) partially w.r. to x ,

$1 \cdot \frac{\partial}{\partial x} + x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$

$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} \quad \dots (2)$

Again,

Diff. (1) partially w.r. to y ,

$\frac{\partial}{\partial y} \left[x \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[y \frac{\partial u}{\partial y} \right] = \frac{\partial u}{\partial y}$

$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y}$

$\Rightarrow xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots (3)$

Adding (2) and (3)

$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0. \quad \text{Ans.}$

Q.3. Attempt any Four parts: (5×4=20)

(a) Expand $e^x \cos y$ about the point $\left(1, \frac{\pi}{4} \right)$.

Ans. Given function

$$\begin{aligned}
f(x, y) &= e^x \cos y \\
\Rightarrow f &= e^x \cos y \\
\Rightarrow f &= -e^x \sin y \\
f_{xx} &= e^x \cos y \\
f_{xy} &= -e^x \sin y \\
f_{yy} &= -e^x \cos y \\
f_{xxx} &= -e^x \cos y \\
f_{xyy} &= -e^x \cos y \\
f_{yyy} &= e^x \sin y
\end{aligned}$$

Thus we get the following table :

Function	Value : at $\left(1, \frac{\pi}{4}\right)$
f	$\frac{e}{\sqrt{2}}$
f_x	$\frac{e}{\sqrt{2}}$
f_y	$-\frac{e}{\sqrt{2}}$
f_{xx}	$\frac{e}{\sqrt{2}}$
f_{xy}	$-\frac{e}{\sqrt{2}}$
f_{yy}	$-\frac{e}{\sqrt{2}}$
f_{xxx}	$\frac{e}{\sqrt{2}}$
f_{xyy}	$-\frac{e}{\sqrt{2}}$
f_{yyy}	$-\frac{e}{\sqrt{2}}$
f_{yyy}	$\frac{e}{\sqrt{2}}$

\therefore By Taylor's theorem,

$$\begin{aligned}
f(x, y) &= f\left(1, \frac{\pi}{4}\right) + \frac{1}{1!} \left[(x-1) f_x + \left(y - \frac{\pi}{4}\right) f_y \right] + \frac{1}{2!} \\
&\quad \left[(x-1)^2 f_{xx} + 2(x-1) \left(y - \frac{\pi}{4}\right) f_{xy} + \left(y - \frac{\pi}{4}\right)^2 f_{yy} \right] \\
&= \frac{e}{\sqrt{2}} + \frac{1}{1!} \left[(x-1) \frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right) \left(-\frac{e}{\sqrt{2}}\right) \right] + \frac{1}{2!}
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{e}{\sqrt{2}} (x-1)^2 - 2 \frac{e}{\sqrt{2}} (x-1) \left(y - \frac{\pi}{4}\right) - \frac{e}{\sqrt{2}} \left(y - \frac{\pi}{4}\right)^2 \right] \\
&+ \frac{1}{3!} \left[\frac{e}{\sqrt{2}} (x-1)^3 - 3 \frac{e}{\sqrt{2}} (x-1)^2 \left(y - \frac{\pi}{4}\right) \right. \\
&\quad \left. - \frac{3e}{\sqrt{2}} (x-1) \left(y - \frac{\pi}{4}\right)^2 + \frac{e}{\sqrt{2}} \left(y - \frac{\pi}{4}\right)^3 \right] \\
&= \frac{e}{\sqrt{2}} \left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{1}{2} (x-1)^2 - (x-1) \right. \\
&\quad \left. \left(y - \frac{\pi}{4}\right) - \frac{1}{2} \left(y - \frac{\pi}{4}\right)^2 + \frac{1}{6} (x-1)^3 \right. \\
&\quad \left. - \frac{1}{2} (x-1) \left(y - \frac{\pi}{4}\right)^2 + \frac{1}{6} \left(y - \frac{\pi}{4}\right)^3 \right] + \dots
\end{aligned}$$

Q. 3 (b) Calculate the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ of the following :

$$\begin{aligned}
u &= x + 2y + z \\
v &= x + 2y + 3z \\
w &= 2x + 3y + 5z
\end{aligned}$$

$$\begin{aligned}
\text{Ans. } \frac{\partial(u, v, w)}{\partial(x, y, z)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} \\
&= 1(1) - 2(-1) + 1(-1) \\
&= 2 + 1 - 1 \\
&= 2. \quad \text{Ans.}
\end{aligned}$$

Q. 3 (c) Discuss the maxima and minima of the function: $f(x, y) = \cos x \cos y \cos(x+y)$

$$\begin{aligned}
\text{Ans. Hence, } f(x, y) &= \cos x \cos y \cos(x+y) \\
&= \cos y [-\sin x \cos(x+y) \\
&\quad - \cos x \sin(x+y)] \\
&= -\cos y [\sin x \cos(x+y) \\
&\quad + \cos x \sin(x+y)] \\
&= -\cos y \sin(x+y) \\
&\quad [\because \sin A \cos B + \cos A \sin B = \sin(A+B)]
\end{aligned}$$

$$\Rightarrow \frac{\partial f}{\partial x} = -\sin(2x+y) \cos y$$

Similarly,

$$\frac{\partial f}{\partial y} = -\sin(x+2) \cos x$$

For stationary values,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$$\text{Now, } \frac{\partial f}{\partial x} = 0,$$

$$\Rightarrow -\sin(2x+y) \cos y = 0$$

$$\Rightarrow \sin(2x+y) = 0$$

$$\Rightarrow \sin(2x+y) = \sin \pi$$

$$\Rightarrow 2x+y = \pi - (1)$$

Similarly,

$$\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow x+2y = \pi$$

Solving (1) and (2),

$$x = y = \frac{\pi}{3}$$

$$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right) \text{ is stationary point.}$$

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = -2 \cos(2x+y) \cos y$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = -\cos(x+2y) \cos x$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -2 \cos(x+2y) \cos x$$

$$\text{At } \left(\frac{\pi}{3}, \frac{\pi}{3}\right),$$

$$r = -2 \cos \pi \cos \frac{\pi}{3} = -2(-1) \left(\frac{1}{2}\right) = 1$$

$$s = -\cos \pi \cos \frac{\pi}{3} = -(-1) \left(\frac{1}{2}\right) = \frac{1}{2}$$

$$t = -2 \cos \pi \cos \frac{\pi}{3} = (-2)(-1) \left(\frac{1}{2}\right) = 1$$

$$\text{Now, } xt - s^2 = (1)(1) - \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} (+ve)$$

$$\text{Also, } r = 1 (+ve)$$

$$\therefore xt - s^2 \text{ is } +ve$$

and r is also $+ve$

$$\therefore f \text{ is minimum at } \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$$

$$\text{Minimum value} = \cos \frac{\pi}{3} \cos \frac{\pi}{3} \cos \frac{2\pi}{3}$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{-1}{2}$$

$$= -\frac{1}{8} \text{ Ans.}$$

Q.3. (d) Find a point on the ellipse $4x^2 + y^2 = 4$ nearest to the point $(1, 2)$.

Ans. Given ellipse is

$$4x^2 + y^2 = 4$$

Let $P(x, y)$ be the point on given ellipse nearest to $Q(1, 2)$

$$\therefore PQ = \sqrt{(x-1)^2 + (y-2)^2}$$

[By distance formula]

$$\text{Let } f = (x-1)^2 + (y-2)^2$$

$$\phi = 4x^2 + y^2 - 4$$

By Lagrange's method,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow 2(x-1) + \lambda(8x) = 0 \quad \dots (1)$$

$$2(y-2) + \lambda(2y) = 0 \quad \dots (2)$$

$$\text{From (1), } x - 1 + 4x\lambda = 0$$

$$\Rightarrow x = \frac{1}{1+4\lambda}$$

$$\text{From (2), } y - 2 + \lambda y = 0$$

$$\Rightarrow y = \frac{2}{1+\lambda}$$

Putting x and y in $4x^2 + y^2 = 4$, we get

$$\frac{4}{(1+4\lambda)^2} + \frac{4}{(1+\lambda)^2} = 4, \text{ we get}$$

Solving above, we shall obtain λ , and then x and y to obtain desired point.

Q.3. (e) Find the extreme value of $x^2 + y^2 + z^2$ subject to the condition $xy + yz + zx = p$.

Ans. Given function

$$f = x^2 + y^2 + z^2$$

condition is $xy + yz + zx = p$

$$\Rightarrow p = xy + yz + zx - p$$

By log range's method,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$$\Rightarrow 2x + \lambda(y + z) = 0 \quad \dots (1)$$

$$2y + \lambda(x + z) = 0 \quad \dots (2)$$

$$2x + \lambda(x + y) = 0 \quad \dots (3)$$

Eqn. (1) - Eqn. (2) gives,

$$x - y + \lambda(y - x) = 0$$

$$x = y$$

Eqn. (2) - Eqn. (3) gives,

$$y - z + \lambda(z - y) = 0$$

$$y = z$$

Thus, $x = y = z$

$$\therefore xy + yz + zx = p$$

$$\Rightarrow 3x^2 = p$$

$$\Rightarrow x^2 = \frac{p}{3}$$

$$\Rightarrow y^2 = \frac{p}{3} = z^2$$

\therefore Extreme value of

$$f = x^2 + y^2 + z^2$$

$$= \frac{p}{3} + \frac{p}{3} + \frac{p}{3} = p. \quad \text{Ans.}$$

Q.3. (f) If $f(x, y) = x^2 y^{1/10}$, compute the value of f when $x = 1.99$ and $y = 3.01$.

Ans. Hence $f = x^2 y^{1/10}$

To calculate $f(1.99, 3.01)$

$$\text{Let } x = 2, \quad y = 3$$

$$\Rightarrow x + \delta x = 1.99, \quad y + \delta y = 3.01$$

$$\Rightarrow \delta x = -0.01, \quad \delta y = +0.01$$

$$\therefore f = x^2 y^{1/10}$$

$$\Rightarrow \delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$$

$$\Rightarrow \delta f = 2xy^{1/10} \delta x + \frac{1}{10} x^2 y^{-9/10} \delta y$$

$$\Rightarrow \delta f = 2(2)(3)^{1/10} \cdot (-0.01) + \frac{1}{10} \cdot (2)^2 (3)^{-9/10} (0.01)$$

$$= 3^{1/10} \left[-0.04 + \frac{1}{10} (4)(3^{-1})(0.01) \right]$$

$$= 3^{1/10} \left[-0.04 + \frac{0.004}{3} \right]$$

$$= 3^{-9/10} [-0.12 + 0.004]$$

$$= 3^{-9/10} [(-0.116)]$$

$$\Rightarrow f(1.99, 3.01) = f(2, 3) + \delta f$$

$$= 2^2 (3)^{1/10} + (-0.116) 3^{-9/10}$$

$$= 3^{1/10} \left[4 - \frac{0.116}{3} \right]$$

$$= 3^{-9/10} [12 - 0.116]$$

$$= 3^{-9/10} \times 11.884$$

$$= 0.3720 \times 11.884 = 4.4213. \quad \text{Ans.}$$

Q.4. Attempt any Four parts: $(5 \times 4 = 20)$

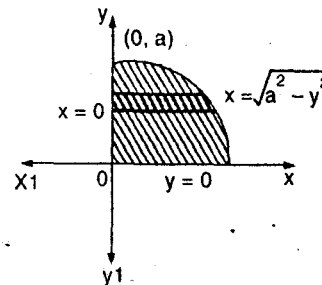
(a) Evaluate the following by changing into polar

$$\text{co-ordinates: } \int_0^a \int_0^{\sqrt{a^2 - y^2}} dy dx.$$

Ans. Given integral is

$$I = \int_{y=0}^a \int_{x=0}^{\sqrt{a^2 - y^2}} y^2 \sqrt{x^2 + y^2} dy dx$$

The region of integration is shown in figure.



Changing the polars.

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\Rightarrow dx dy = r dr d\theta$$

Also, limits are $0 \leq r \leq a, 0 \leq \theta \leq \frac{\pi}{2}$

$$\Rightarrow I = \int_{\theta=0}^{\pi/2} \int_{r=0}^a (r \sin \theta)^2 \sqrt{r^2} r dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^a r^4 \sin^2 \theta dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^5}{5} \right]_{r=0}^a \sin^2 \theta d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{a^5}{5} \sin^2 \theta d\theta$$

$$= \frac{a^5}{5} \int_{\theta=0}^{\pi/2} \sin^2 \theta d\theta$$

$$= \frac{a^5}{5} \int_{\theta=0}^{\pi/2} \sin^2 \theta \cos \alpha \theta d\theta$$

$$= \frac{a^5}{5} \left[\frac{\sqrt{\frac{2+1}{2}} \sqrt{\frac{0+1}{2}}}{2\sqrt{\frac{2+0+2}{2}}} \right]$$

$$\therefore \int_0^{\pi/2} \sin^2 \theta \cos \alpha \theta d\theta = \frac{\sqrt{\frac{p+1}{2}} \sqrt{\frac{q+1}{2}}}{2\sqrt{\frac{p+q+2}{2}}}$$

$$= \frac{a^5}{5} \cdot \frac{\sqrt{\frac{3}{2}} \sqrt{\frac{1}{2}}}{2 \cdot \sqrt{2}} = \frac{a^5}{5} \cdot \frac{\frac{1}{2} \sqrt{\pi} \sqrt{\pi}}{1} = \frac{\pi a^5}{20} \text{ Ans.}$$

Q.4 (b) Find the area enclosed between the parabola $y = 4x - x^2$ and the line $y = x$.

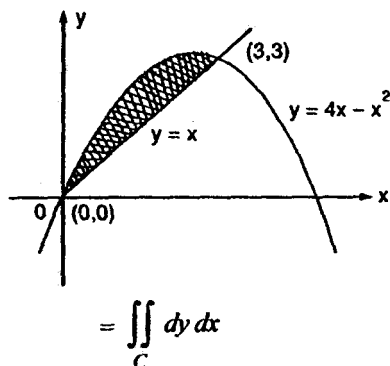
Ans. Given parabola is $y = 4x - x^2$

Given line is $y = x$

$$\begin{aligned} \text{Now, } y &= 4x - x^2 \\ &= -(x^2 - 4x) \\ &= -[(x-2)^2 - 4] \end{aligned}$$

$$\Rightarrow y - 4 = -(x-2)^2$$

Required area = area of shaded region R



Limits are, $x \leq y \leq 4x - x^2$
 $0 \leq x \leq 3$.

$$\Rightarrow \text{Required area} = \int_{x=0}^3 \int_{y=x}^{4x-x^2} dy dx$$

$$= \int_{x=0}^3 [y]_{y=x}^{4x-x^2} dx$$

$$= \int_{x=0}^3 (4x - x^2 - x) dx$$

$$= \int_0^3 (3x - x^2) dx = \left[3 \frac{x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$$= \frac{27}{2} - 9 = \frac{9}{2} \text{ Ans.}$$

Q.4 (c) Change the order of integration in

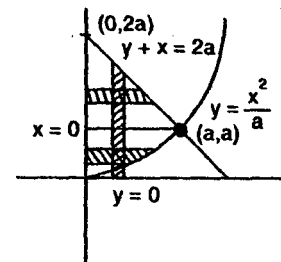
$$\int_0^a \int_{x^2/a}^{2a-x} f(x,y) dx dy.$$

$$\text{Ans. Given integral } I = \int_{x=0}^a \int_{y=x^2/a}^{2a-x} f(x,y) dx dy$$

Region of integration is shown in the figure by shaded portion.

By changing the order of integration

$$I = \iint_{R_1} f(x,y) dx dy + \iint_{R_2} f(x,y) dx dy$$



$$\text{For } R_1, \quad 0 \leq x \leq \sqrt{ay}$$

$$0 \leq y \leq a$$

$$\text{For } R_2, \quad 0 \leq x \leq 2a - y$$

$$a \leq y \leq 2a$$

$$\Rightarrow I = \int_{y=0}^a \int_{x=0}^{\sqrt{ay}} f(x,y) dx dy$$

$$+ \int_{x=0}^{2a} \int_{y=0}^{2a-x} f(x,y) dx dy. \text{ Ans.}$$

Q.4. (d) Find the volume of the solid surrounded by

the surface $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$

Ans. Given surface is

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

$$\text{Required volume} = 3 \iiint_{V'} dx dy dz$$

where V' is the region,

$$x \geq 0, y \geq 0, z \geq 0$$

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} < 1$$

$$\text{Put } \left(\frac{x}{a}\right)^{2/3} = X, \left(\frac{y}{b}\right)^{2/3} = Y, \left(\frac{z}{c}\right)^{2/3} = Z,$$

$$\Rightarrow \frac{x}{a} = X^{3/2}, \frac{y}{b} = Y^{3/2}, \frac{z}{c} = Z^{3/2}$$

$$\Rightarrow x = aX^{3/2}, y = bY^{3/2}, z = cZ^{3/2}$$

$$\Rightarrow dx = \frac{3a}{2} X^{1/2} dX,$$

$$dy = \frac{3b}{2} Y^{1/2} dY, dz = \frac{3c}{2} Z^{1/2} dZ$$

$$\text{Required volume} = 3 \iiint_{V'} \frac{3a}{2}, \frac{3b}{2}, \frac{3c}{2}$$

$$X^{1/2} Y^{1/2} Z^{1/2} dX dY dZ$$

where V' is the region $X \geq 0, Y \geq 0, Z \geq 0$

$$X + Y + Z \leq 1$$

\Rightarrow Required volume

$$= 27abc \iiint_{V'} X^{3/2-1} Y^{3/2-1} Z^{3/2-1} dX dY dZ$$

$$= 27abc \frac{\sqrt{\frac{3}{2}} \sqrt{\frac{3}{2}} \sqrt{\frac{3}{2}}}{\sqrt{\frac{3}{2} + \frac{3}{2} + \frac{3}{2} + 1}}$$

[By Dirichlet's Theorem]

$$= 27abc \frac{\frac{1}{2} \sqrt{\pi} \frac{1}{2} \sqrt{\pi} \frac{1}{2} \sqrt{\pi}}{\sqrt{\frac{11}{2}}}$$

$$= \frac{27abc}{2} \frac{\pi}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$= X = \frac{4abc\pi}{35}. \text{ Ans.}$$

Q.4. (e) Define Gamma and Beta functions. Prove that $B(l, m) B(l+m, n) B(l+m+n, p)$

$$= \frac{[l][m][n][p]}{[l+m+n+p]}$$

Ans. Gamma Function : Gamma function denoted by Γn is defined as

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx \quad n > 0$$

Beta Function : Beta function is denoted by $B(m, n)$ and is defined as

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

2nd Part : To prove that

$$B(l, m) B(l+m, n) B(l+m+n, p)$$

$$= \frac{[l][m][n][p]}{[l+m+n+p]}$$

$$\text{L.H.S} = B(l, m) B(l+m, n) B(l+m+n, p)$$

$$= \frac{[e][m]}{[e+m]} \cdot \frac{[e+m][n]}{[e+m+n]} \cdot \frac{[e+m+n][p]}{[e+m+n+p]}$$

$$\left[\because B(p, q) = \frac{[p][q]}{[p+q]} \right]$$

$$= \frac{[e][m][n][p]}{[l+m+n+p]} = \text{R.H.S.}$$

$$\text{Q.4. (f) Show that } \int_0^1 x^5 (1-x^3)^{10} dx = \frac{1}{396}$$

$$\text{Ans. Put } I = \int_0^1 x^5 (1-x^3)^{10} dx$$

$$\text{Put } x^3 = y$$

$$\Rightarrow x = y^{\frac{1}{3}}$$

$$\Rightarrow dx = \frac{1}{3} y^{-\frac{2}{3}} dy$$

$$\text{At } x=0, y=0$$

$$\text{At } x=1, y=1$$

$$\Rightarrow I = \int_0^1 y^{\frac{5}{3}} (1-y)^{10} \cdot \frac{1}{3} y^{-\frac{2}{3}} dy$$

$$= \frac{1}{3} \int_0^1 y (1-y)^{10} dy$$

$$= \frac{1}{3} \int_0^1 y^{2-1} (1-y)^{11-1} dy$$

$$= \frac{1}{3} B(2, 11)$$

$$\left[\because \int_0^1 x^{m-1} (1-x)^{n-1} dx = B(m, n) \right]$$

$$= \frac{1}{3} \frac{[2][11]}{[2+11]}$$

$$= \frac{1! \times 10!}{3 \times 12!}$$

$$= \frac{1}{3} \frac{1! \times 10!}{12 \times 11 \times 10!}$$

$$= \frac{1}{3 \times 11 \times 12}$$

$$= \frac{1}{396}$$

Q. 5. Attempt any Three parts : (5×4=20)

(a) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then show that

(i) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$, where \vec{a} is a constant vector.

(ii) $\text{grad } r = \frac{\vec{r}}{r}$

(iii) $\text{grad } \frac{1}{r} = -\frac{\vec{r}}{r^3}$

(iv) $\text{grad } r^n = n r^{n-2} \vec{r}$ where $r = |\vec{r}|$

$$\text{Ans. (i)} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\text{Let } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\Rightarrow \vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\Rightarrow \nabla(\vec{a} \cdot \vec{r}) = \hat{i} \frac{\partial}{\partial x} (a_1x + a_2y + a_3z)$$

$$+ \hat{j} \frac{\partial}{\partial y} (a_1x + a_2y + a_3z) + \hat{k} \frac{\partial}{\partial z} (a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$= \vec{a} \quad \left[\because \text{grad } \phi = \sum \frac{i \partial \phi}{\partial x} \right]$$

Before Proving (ii) and (iii), we shall prove (iv)

$$\text{i.e., } \text{grad } r^n = n r^{n-2} \vec{r} \quad n\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow r = |\vec{r}|$$

$$\Rightarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\text{Now, } \text{grad } f = \sum i \frac{\partial f}{\partial x}$$

$$\Rightarrow \text{grad } r^n = \sum i \frac{\partial}{\partial r} (r^n) \cdot \frac{\partial r}{\partial x}$$

$$\begin{aligned} \Rightarrow \text{grad } r^n &= \sum i (n r^{n-1}) \frac{x}{r} \\ &= \sum n r^{n-2} x i = n r^{n-2} \sum x i \\ &= \sum n r^{n-2} (x\hat{i} + y\hat{j} + z\hat{k}) \end{aligned}$$

$$\Rightarrow \text{grad } r^n = n r^{n-2} \vec{r} \quad \dots (2)$$

Hence Proved.

(ii) To prove $\text{grad } r = \frac{\vec{r}}{r}$

Putting $n=1$ in (2),

$$\text{grad } r^1 = 1 r^{1-2} \vec{r}$$

$$\Rightarrow \text{grad } r = \frac{\vec{r}}{r}$$

(iii) To prove $\text{grad } \frac{1}{r} = -\frac{\vec{r}}{r^3}$

Putting $n=-1$ in (2), we get

$$\text{grad } r^{-1} = (-1) r^{-1-2} \vec{r}$$

$$\Rightarrow \text{grad } \frac{1}{r} = -\frac{\vec{r}}{r^3}$$

Q.5. (b) Prove that $\vec{a} \times (\nabla \times \vec{r}) = \nabla (\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla) \vec{r}$
 $-(\vec{a} \cdot \nabla) \vec{r}$ where \vec{a} is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Ans. To prove that

$$\vec{a} \times (\nabla \times \vec{r}) = \nabla (\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla) \vec{r}$$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Now,

$$(\nabla \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= 0$$

$$\Rightarrow \nabla \times \vec{r} = 0$$

$$\Rightarrow \vec{a} \times (\nabla \times \vec{r}) = 0 \quad \dots(1)$$

Now

$$\vec{a} \cdot \vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow \vec{a} \cdot \vec{r} = a_1x + a_2y + a_3z$$

$$\Rightarrow \nabla (\vec{a} \cdot \vec{r}) = \hat{i} \frac{\partial}{\partial x} (a_1x + a_2y + a_3z)$$

$$+ \hat{j} \frac{\partial}{\partial y} (a_1x + a_2y + a_3z)$$

$$+ \hat{k} \frac{\partial}{\partial z} (a_1x + a_2y + a_3z)$$

$$= \hat{i} a_1 + \hat{j} a_2 + \hat{k} a_3$$

$$\Rightarrow \nabla (\vec{a} \cdot \vec{r}) = \vec{a} \quad \dots(2)$$

$$(\vec{a} \cdot \nabla) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot$$

$$= a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$$

$$\Rightarrow (\vec{a} \cdot \nabla) \vec{r} = \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z} \right)$$

$$(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$= \vec{a}$$

....(3)

From (2) and (3)

$$\nabla (\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla) \vec{r} = \vec{a} - \vec{a}$$

$$\Rightarrow \nabla (\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla) \vec{r} = 0 \quad \dots(4)$$

From (1) and (4),

$$\vec{a} \times (\nabla \times \vec{r}) = \nabla (\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \nabla) \vec{r}$$

Q.5. (c) State the Green's theorem. Verify it by

evaluating $\oint_C [(x^3 - xy^3)dx + (y^2 - 2xy)dy]$ where

C is the square having the vertices at the points $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$.

Ans. Green's Theorem : Green's theorem states that

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

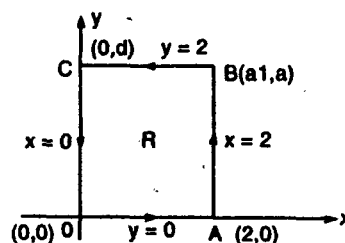
where C is a closed curve bounding the region R .

2nd Part : Given integral

$$I = \oint_C [x^3 - xy^3 + dx + (y^2 - 2xy) du]$$

$$= \oint_C [Pdx + Qdy]$$

where C is the square shown in figure.



Now, to verify Green's theorem we have to show,

$$\begin{aligned} \int_C (Pdx + Qdy) &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &+ \int_{AB} (Pdx + Qdy) + \int_{BC} (Pdx + Qdy) \\ &+ \int_{CO} (Pdx + Qdy) \quad \dots (1) \end{aligned}$$

Along OA: $y=0 \Rightarrow dy=0$
 x varies from 0 to 2

$$\therefore \int_{OA} Pdx + Qdy = \int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 = 4 \quad \dots (2)$$

Along AB: $x=2 \Rightarrow dx=0$
 y varies from 0 to 2

$$\begin{aligned} \therefore \int_{AO} (Pdx + Qdy) &= \int_0^2 (y^3 - 4y) dy \\ &= \left[\frac{y^4}{4} - 2y^2 \right]_0^2 \\ &= \frac{8}{4} - 8 = -\frac{16}{4} \quad \dots (3) \end{aligned}$$

Along BC, $y=2, \Rightarrow dy=0$
 x varies from 2 to 0.

$$\begin{aligned} \therefore \int_{BC} (Pdx + Qdy) &= \int_2^0 (x^3 - 8x) dx \\ &= \left[\frac{x^4}{4} - \frac{8x^2}{2} \right]_2^0 = \left(\frac{x^4}{4} - 4x^2 \right) \Big|_2^0 \\ &= -\frac{16}{4} + 16 = 12 \quad \dots (4) \end{aligned}$$

Along CO, $x=0, dx=0$
 y varies from 2 to 0

$$\therefore \int_{CO} Pdx + Qdy = \int_2^0 y^2 dy$$

$$= \left[\frac{y^3}{3} \right]_2^0 = -\frac{8}{3} \quad \dots (5)$$

Putting values from (2), (3), (4) and (5), (1) gives

$$\begin{aligned} \int_C Pdx + Qdy &= 4 - \frac{16}{3} + 12 - \frac{8}{3} \\ \Rightarrow \int_C Pdx + Qdy &= 8 \quad \dots (6) \end{aligned}$$

$$\begin{aligned} \text{Again, } \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} (y^2 - 2xy) = -2y \\ \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} (x^3 - xy^3) = -3xy^2 \end{aligned}$$

$$\therefore \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \iint_R (-2y + 3xy^2) dx dy$$

Limits are, $0 \leq x \leq 2$
 $0 \leq y \leq 2$

$$\begin{aligned} \Rightarrow \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy &= \int_{x=0}^2 \int_{y=0}^2 (-2y + 3xy^2) dy dx \\ &= \int_{x=0}^2 \left[-y^2 + 3x \frac{y^3}{3} \right]_{y=0}^2 dx \\ &= \int_{x=0}^2 (-4x + 4x^2) dx \\ &= \left[-4x + 4x^2 \right]_0^2 = -8 + 16 = 8 \quad \dots (7) \end{aligned}$$

From (6) and (7)

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Hence Green's theorem is verified.